Whitham's highest wave: convexity and uniqueness

Special Session on Advances in the modeling of fluid-structure interaction systems and computer-assisted rigorous bounds
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Damien Galant

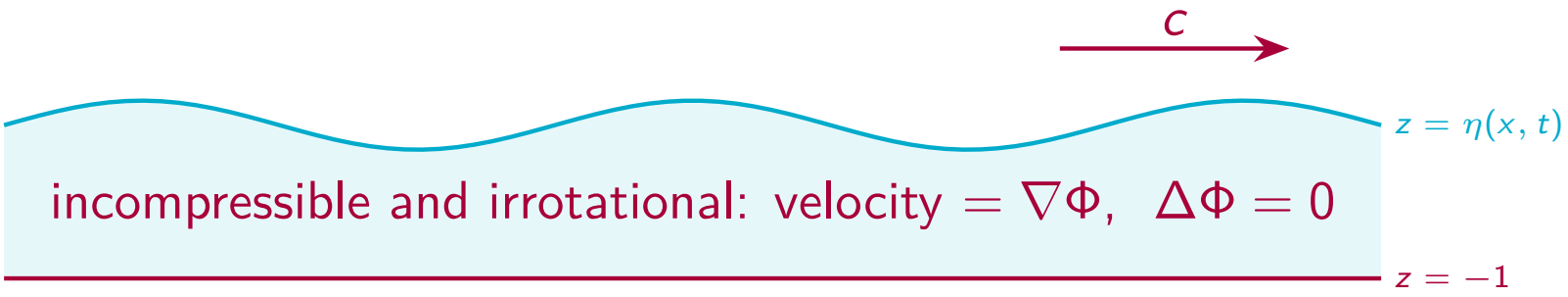
Department of Mathematics
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Joint work with Alberto Enciso, Javier Gómez-Serrano and Bruno Vergara

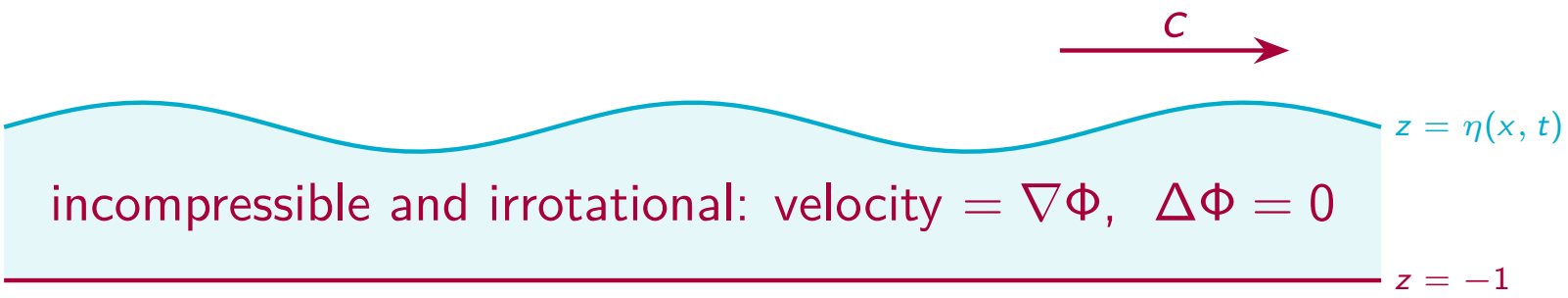
Sunday 11 October 2026

Gravity water waves



Flat bed, gravity, no surface tension; depth and g equal to one. A *Stokes wave* is a periodic traveling wave of this nonlinear problem; the *highest* one has a stagnation point at its crest.

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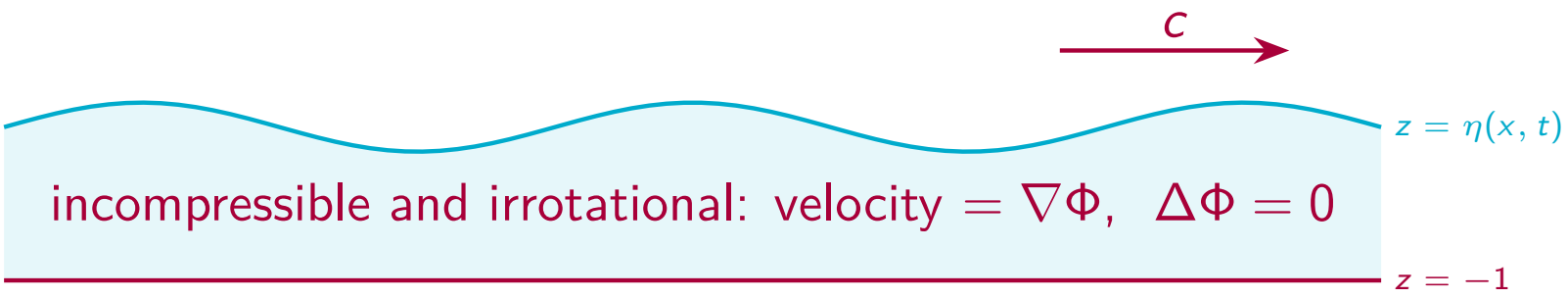
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Linearize about still water. With η the elevation and Φ the velocity potential,

$$\Phi_{xx} + \Phi_{zz} = 0 \quad (-1 < z < 0), \qquad \Phi_z = 0 \quad (z = -1),$$

$$\underbrace{\eta_t = \Phi_z}_{\text{kinematic}}, \qquad \underbrace{\Phi_t + \eta = 0}_{\text{Bernoulli}} \qquad (z = 0).$$

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The surface moves with the fluid; the pressure there is atmospheric.

The dispersion relation of water waves

Fourier transform in x, with $k := |\xi|$ and $q := \Phi|_{z=0}$. Laplace's equation and the bed condition give

$$\widehat{\Phi}(\xi, z, t) = \widehat{q}(\xi, t)\, \frac{\cosh\big(k(z+1)\big)}{\cosh k}, \qquad \text{so} \qquad \widehat{\Phi_z}\big|_{z=0} = k \tanh k\ \widehat{q}.$$

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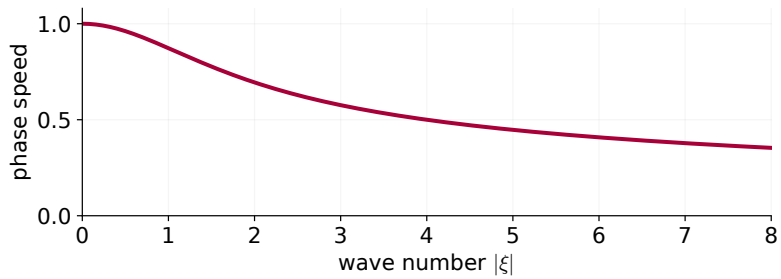
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Right-going waves: $\omega = \xi\, m(\xi)$, with the *phase speed*

$$\boxed{m(\xi) = \sqrt{\frac{\tanh \xi}{\xi}}} \qquad m(0) = 1.$$



Short waves are slow: $m(\xi) \sim |\xi|^{-1/2}$.

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Whitham's motivation

With the correct short waves, the model should allow *wave breaking* (Hur 2017) and a *highest wave with a sharp crest*, as for Euler. KdV has neither.

Traveling waves

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Since $L1 = 1$, the map $v \mapsto v(x - 2at, t) + a$ sends solutions to solutions: shifting φ by a and c by $2a$, we may take $B = 0$ and call μ the speed of the normalized wave: $L\varphi - \mu\varphi + \varphi^2 = 0$.

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Theorem (Ehrnström–Wahlén 2019, by global bifurcation)

Solutions with $\max \varphi < \mu/2$ are smooth. There is an even solution, decreasing on $(0, \pi)$, with $\varphi(0) = \mu/2$: smooth away from the crest and *exactly* $C^{1/2}$ at the crest.

Why a square-root cusp?

Set $u := \mu/2 - \varphi \geq 0$, the depth below the crest. Evaluating the equation at the crest, where $u = 0$, the speed drops out:

$$\boxed{u^2 = Lu - Lu(0) =: \mathcal{L}u} \qquad \text{and} \qquad \mu = 1 - \sqrt{1 - 4\,Lu(0)}.$$

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L is a convolution with a positive, even kernel $K(x) \sim (2\pi|x|)^{-1/2}$ at zero. If $u \sim \lambda|x|^{1/2}$ near the crest, the equation *forces* $\lambda = \sqrt{\pi/8}$, as Whitham argued in 1967:

$$\varphi(x) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}}\,|x|^{1/2} + \dots$$

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Stokes wave: a corner



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Goal: the Whitham counterpart of the Stokes-wave theorems, by a very different route.

Main theorem

Theorem

(i) *Convexity. There is a 2π-periodic traveling wave of extreme form φ_*, with speed $\mu_* \in (0.7666, 0.7700)$, even, decreasing on $(0, \pi)$, convex between consecutive crests, and*

$$\varphi_*(x) = \frac{\mu_*}{2} - \sqrt{\frac{\pi}{8}}\,|x|^{1/2} + O(|x|^{1+\eta}), \qquad \eta > 0.$$

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(ii) *Uniqueness. The monotone 2π-periodic weak traveling waves of extreme form are exactly*

$$v(x,t) = \varphi_*(x - ct - x_0) + \frac{c - \mu_*}{2}, \qquad c \in \mathbb{R}, \quad x_0 \in \mathbb{T}.$$

In particular: one profile per speed, up to translation.

Why monotone? Why weak?

- *Monotone*: non-increasing on $(0, \pi)$ and non-decreasing on $(-\pi, 0)$, after a translation. Some such hypothesis is needed: k-fold copies of $2\pi/k$-periodic waves are 2π-periodic too.

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- *Weak* because we can: every nonconstant monotone traveling wave $\varphi \in L^2(\mathbb{T})$ is continuous, even, strictly monotone and $\leq \mu/2$. By Ehrnström–Wahlén, either it is smooth, or $\varphi(0) = \mu/2$ and it has an exact $C^{1/2}$ cusp.

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"Of extreme form" simply means "not smooth".

Why is it hard?

- *No complex analysis.* Stokes waves live on a harmonic function in a strip; Plotnikov–Toland's convexity proof is a tour de force of complex analysis. Whitham's equation is nonlocal, with no such structure.

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What we do instead

Construct the wave *directly*: a fixed point around an explicit cusped approximation, with estimates sharp enough to control two derivatives, and a computer-assisted proof.

An approximation with the cusp built in

Trigonometric polynomials cannot see $|x|^{1/2}$, and powers $|x|^s$ are not periodic. *Clausen functions* do both jobs: for $1 < s < 3$,

$$C_s(x) := \sum_{n \geq 1} \frac{\cos(nx)}{n^s} = \zeta(s) + \Gamma(1-s) \sin \tfrac{\pi s}{2} \, |x|^{s-1} + O(x^2),$$

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$$u_0(x) = \sum_{s\in S} a_s\big[\zeta(s) - C_s(x)\big] + \sum_{1\leq n\leq 11} b_n\big[\cos(nx) - 1\big],$$

8 exponents $s = \frac{3}{2} + kp_0 + jp_1$, with $a_{3/2} = \frac{1}{4}$ so that $u_0 = \sqrt{\pi/8}\,|x|^{1/2} + \dots$

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$p_0 \approx 0.611$ and $p_1 \approx 2.762$ are *not fitted*: the next terms of the cusp expansion force them. The b_n improve u_0 away from the crest.

From the approximation to an exact wave

Look for $u = u_0 + |x|\, w$ with $w \in L^\infty$ even. Then $u^2 = \mathcal{L}u$ becomes

$$w = (I - T_0)^{-1}\Big(F_0 - \frac{|x|}{2u_0}\, w^2\Big), \qquad F_0 := \frac{\mathcal{L}u_0 - u_0^2}{2|x|u_0},$$

where T_0 is the linearization of $\mathcal{L}$ at u_0, an explicit integral operator.

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where T_0 is the linearization of $\mathcal{L}$ at u_0, an explicit integral operator. Banach's fixed point theorem needs three numbers,

$$\|F_0\|_\infty \leq 10^{-7}, \qquad \|T_0\|_{L^\infty \to L^\infty} \leq 0.998, \qquad \Big\|\frac{x}{2u_0}\Big\|_\infty \leq 2.7,$$

and gives a contraction on the ball of radius $6 \cdot 10^{-5}$:

$$\boxed{|u_*(x) - u_0(x)| \leq 6 \cdot 10^{-5}\, |x|.}$$

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No room to spare

Near the crest the kernel is self-similar and $\|T_0\|$ is governed by an explicit constant $C_B = 0.99736\ldots$ The margin below 1 is built into the problem.

Convexity: two more fixed points

Convexity of $\varphi_* = \mu_*/2 - u_*$ means $u_*'' < 0$ on $(0, \pi]$: two derivatives of a function which is only $C^{1/2}$ at the crest, and a picture that looks convex is not a proof.

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Differentiate the equation once and twice: the errors $u_*' - u_0'$ and $x\,(u_*'' - u_0'')$ solve fixed-point problems of the same type, with more singular kernels but still contracting. The weight x is what makes the second one work up to the cusp.

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Theorem, part (i)

u_* is strictly concave: φ_* is strictly convex between consecutive crests.

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Let $u = \mu/2 - \varphi$ for *any* normalized monotone extreme wave. All we use: u is even, continuous, increasing on $(0, \pi)$, $u(0) = 0$, $u = O(\sqrt{x})$, and

$$u(x)^2 = \int_0^\pi G_x(y)\, u(y)\, \mathrm{d}y, \qquad G_x(y) = K(x-y) + K(x+y) - 2K(y).$$

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The kernel changes sign exactly once: at some $\eta_x \in (0, x)$, with $G_x < 0$ before and $G_x > 0$ after. To bound $u(x)^2$ from below, we need an *upper* bound on u where $G_x < 0$ and a *lower* bound on u where $G_x > 0$.

Trap every competitor

The fixed point gives uniqueness only near u_0: a second extreme wave could be anywhere.

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Plan: start from crude bounds valid for every competitor and let the equation improve them. This is Kobayashi's strategy for Stokes waves.

The iterative scheme

- *Seeds:* $u \leq \min\{\sqrt{x}, 1\}$, and an explicit positive lower bound from integrating the equation by parts against the positive measure $\mathrm{d}u$.

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- *Update* at each node, splitting the kernel by sign:

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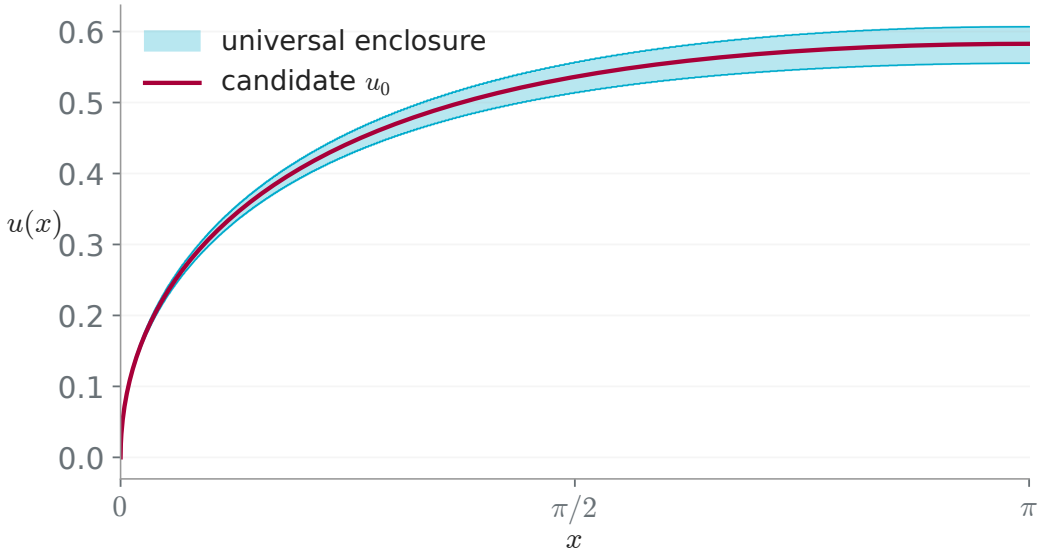
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- *Iterate:* 240 sweeps over 720 nodes, all integrals enclosed by interval arithmetic. Every step keeps a valid enclosure, so the output bounds *every* competitor.

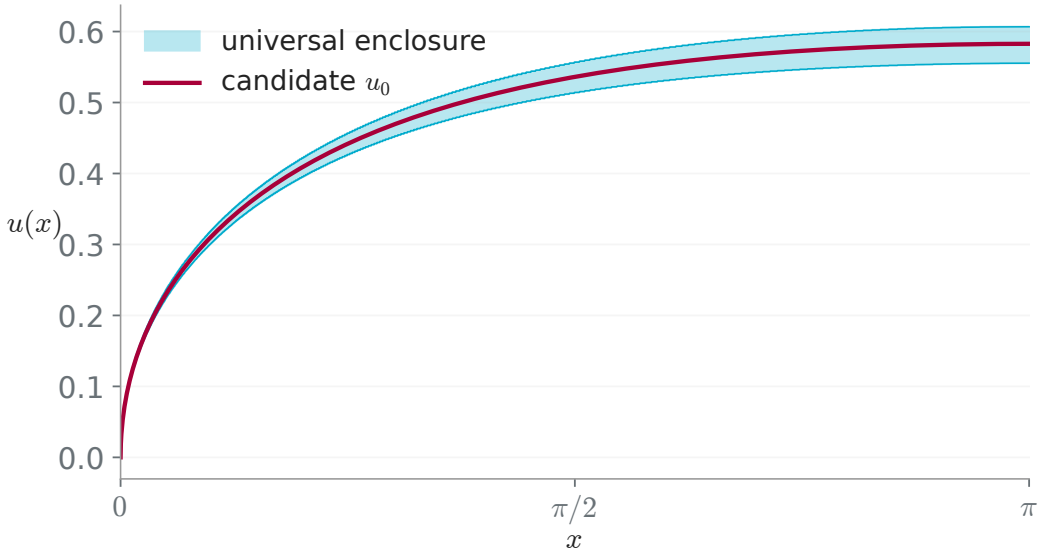
Universal bounds



Result: *every* competitor satisfies

$$0.6105\sqrt{x} < u(x) < 0.6399\sqrt{x} \quad \text{near } 0, \qquad 0.5554 < u(\pi) < 0.6068.$$

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No cusp coefficient is assumed for the competitor: only $u = O(\sqrt{x})$.

The comparison: the key inequality

Let u be any admissible profile, u_* ours, and $d := u - u_*$. Subtracting the reduced equations $u^2 = \mathcal{L}u$ and $u_*^2 = \mathcal{L}u_*$,

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Divide by $u + u_* \geq b + b_* > 0$, where $b \leq u$ is the universal lower bound and $b_* \leq u_*$ comes from the construction, and use $|d(y)| \leq D\,\rho(y)$ inside the integral:

$$\frac{|d(x)|}{\rho(x)} \leq \underbrace{\frac{\int_0^\pi |G_x(y)|\,\rho(y)\,\mathrm{d}y}{\rho(x)\,(b(x) + b_*(x))}}_{=:\,Q(x)}\, D \qquad (0 < x \leq \pi).$$

The comparison: conclusion

Taking the supremum over x: $D \leq Q\,D$, with $Q := \sup_{0<x\leq\pi} Q(x)$.

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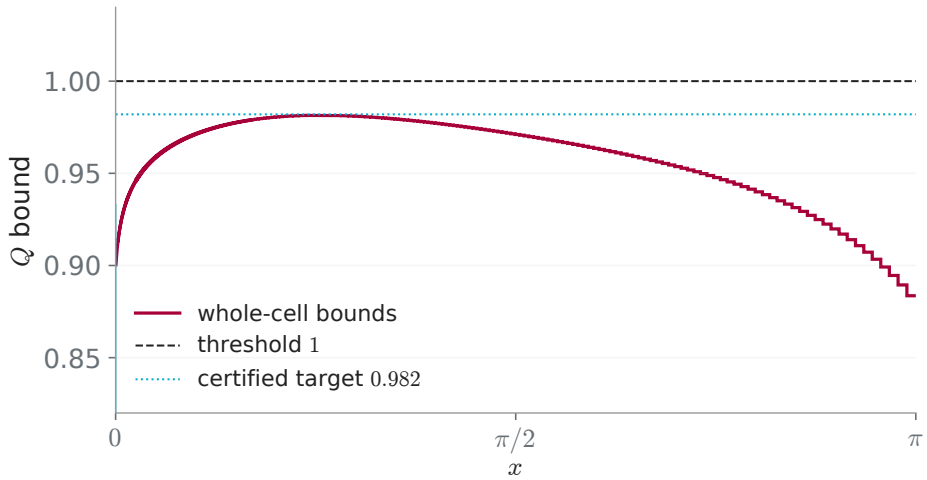
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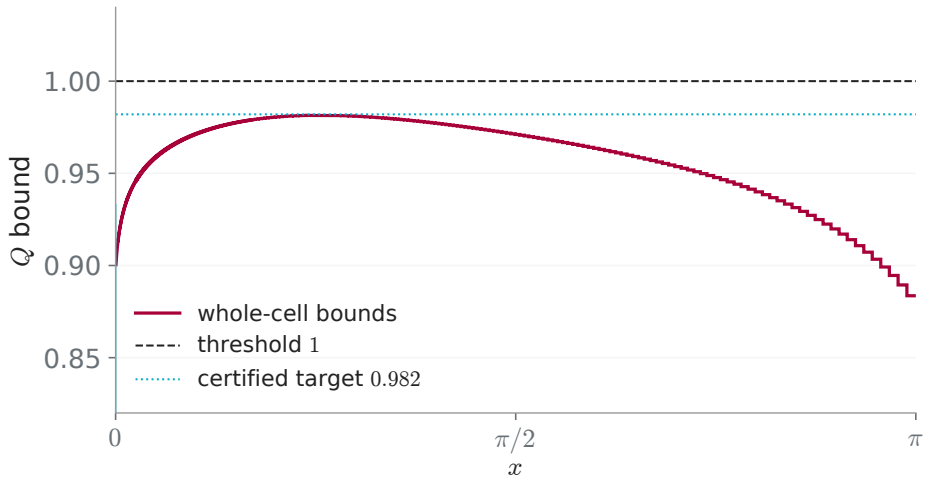


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Hence $D = 0$, that is, $u = u_*$.

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Certified on the cusp interval and on each of the 719 cells: $Q < 0.982$. Hence $D = 0$, that is, $u = u_*$. Recover μ from $Lu(0)$ and undo the normalization: Theorem, part (ii).

Take-home message

Main message

Whitham's highest wave is *unique*, *convex* and has a *square-root cusp*: the counterpart, for Whitham's equation, of the Stokes-wave theorems, proved by a direct construction, universal a priori bounds and computer-assisted estimates.

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What the computer does: interval arithmetic (Arb) on explicit integrals, giving bounds on *whole intervals*, never samples. Analysis handles the cusp; the computer handles the rest.

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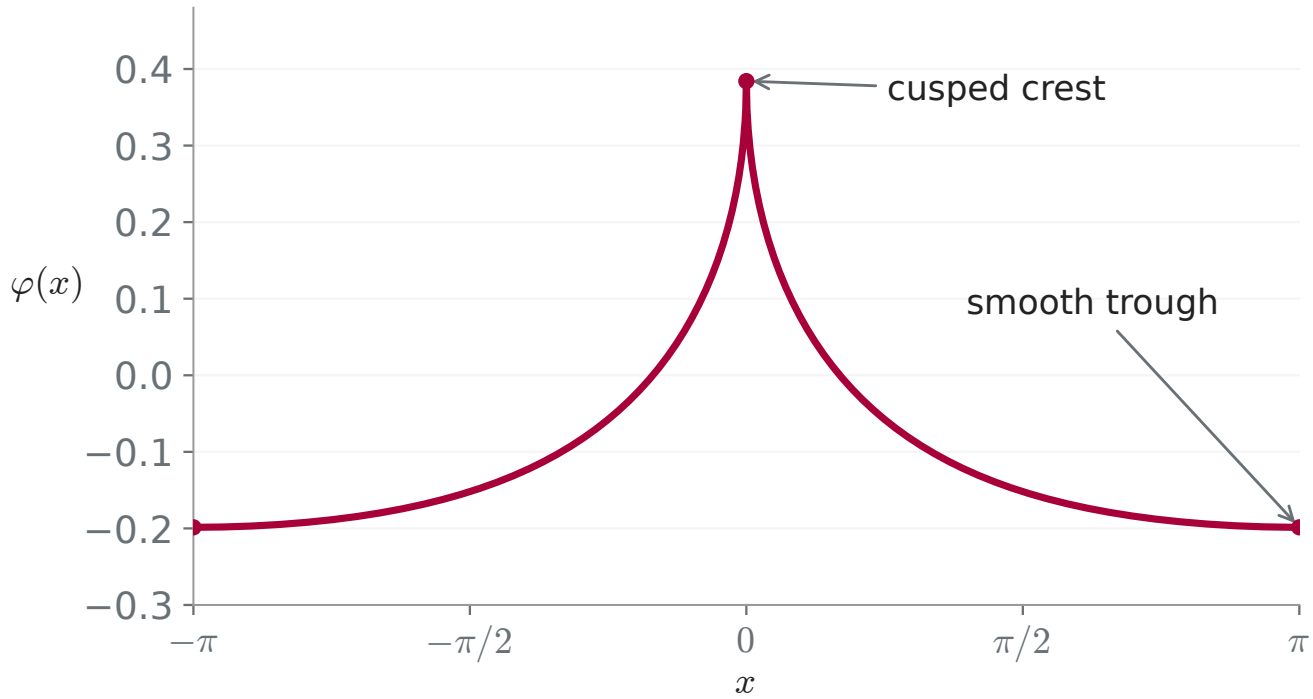
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Perspectives:

- other periods than 2π?
- non-symmetric waves of extreme form, as expected for Euler?
- other dispersions of Whitham type?

Thank you!


0.4
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
$\varphi(x)$
cusped crest
smooth trough
$-\pi$
$-\pi/2$
0
$\pi/2$
π
x

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This talk is based on

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 *Uniqueness, convexity and sharp asymptotics for Whitham's highest wave.* In preparation.

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The kernel of L

For $x \in [-\pi, \pi]$, $Lf(x) = \int_{-\pi}^{\pi} K(y)\, f(x-y)\, \mathrm{d}y$ with

$$K(x) = \sum_{n \geq 1} \int_{(n-\frac{1}{2})\pi}^{n\pi} \frac{\cosh\big[s(\pi - |x|)\big]}{\pi \sinh(s\pi)} \left(\frac{|\tan s|}{s}\right)^{1/2} \mathrm{d}s.$$

K is positive, even, decreasing on $(0, \pi)$, with unit mass, and

$$K(x) = \frac{1}{\sqrt{2\pi|x|}} + R(x), \qquad R \in C^2 \text{ near } 0.$$

For the reduced equation, only $G_x(y) = K(x-y) + K(x+y) - 2K(y)$ enters. It is singular at $y = 0$ and $y = x$, and changes sign exactly once on $(0, x)$.

The lower seed

Let J_x be the primitive of G_x with $J_x(0) = J_x(\pi) = 0$. Integrating by parts against the positive measure $\mathrm{d}u$,

$$u(x)^2 = \int_0^\pi G_x\, u\, \mathrm{d}y = -\int_0^\pi J_x\, \mathrm{d}u.$$

For $0 < z < x$, set $H_{x,z} := J_z - J_x$. Then

$$u(x)^2 - u(z)^2 = \int_0^\pi H_{x,z}(y)\, \mathrm{d}u(y),$$

and $H_{x,z} > 0$ on $(0, \pi)$, concave on $[z, x]$ (since K is decreasing and K' increasing). Keeping only $y \in [z, x]$,

$$(u(x) - u(z))(u(x) + u(z)) \geq \min\{H_{x,z}(z), H_{x,z}(x)\}\,(u(x) - u(z)),$$

hence $u(x) \geq \frac{1}{2} \min\{H_{x,z}(z), H_{x,z}(x)\}$. The computation uses $z = 0.437\, x$.

The operator norm of T_0

$$T_0 w(x) = \frac{1}{2x u_0(x)} \int_0^\pi G_x(y)\, y\, w(y)\, \mathrm{d}y, \quad \|T_0\| = \sup_{0<x\le\pi} \frac{\int_0^\pi |G_x(y)|\, y\, \mathrm{d}y}{2x u_0(x)}.$$

Near the crest: rescale $y = xt$ and write $K(z) = (2\pi|z|)^{-1/2} + R(z)$. The homogeneous part gives

$$C_B = \frac{1}{\pi} \int_0^\infty \left| \frac{1}{\sqrt{|1-t|}} + \frac{1}{\sqrt{1+t}} - 2 \right| t^{-5/2}\, \mathrm{d}t = 0.997361598\ldots,$$

and R is bounded analytically.

Away from the crest: enclose the zero of G_x, integrate each sign region with explicit primitives of K, and certify *whole intervals* of x with interval arithmetic.

Common bound for T_0, T_1, T_2: 0.998. Neumann series: $\|(I - T_0)^{-1}\| \le 500$.